Derivation of cost functions for Leontief technology

Derive the cost functions for the following technologies:

(a) Let $f(x_1, x_2)$ be the production function defined by $f(x_1, x_2) = \min\{ax_1, bx_2\}$.

(b) For the production function $f(x_1, x_2, \ldots, x_n)$, it is given by $f(x_1, x_2, \ldots, x_n) = \min\{x_1, x_2, \ldots, x_n\}$.

(c) The production function $f(x_1, x_2, \ldots, x_n)$ is described by $f(x_1, x_2, \ldots, x_n) = \min\{a_1x_1, a_2x_2, \ldots, a_nx_n\}$.

Solutions

1. The problem to be solved is to minimize the costs $p_1x_1 + p_2x_2$ subject to a certain desired production level $\bar{Q}$. We can convert this two-variable problem into a single-variable problem. We know that at the optimum, it will always be true that:

$$ax_1 = bx_2$$

Otherwise, it would mean that more inputs are being used for the same amount of product, which would not be optimal. Using this fact in the production function:

$$\bar{Q} = \min\{ax_1, bx_2\} = \min\{ax_1, ax_1\} = ax_1$$

And also:

$$\bar{Q} = bx_2$$

Therefore:

$$x_1 = \frac{\bar{Q}}{a}$$

and

$$x_2 = \frac{\bar{Q}}{b}$$

Replacing in the function to minimize:

$$C = p_1\frac{\bar{Q}}{a} + p_2\frac{\bar{Q}}{b}$$

Thus, the cost function is:

$$C = \bar{Q}\left(\frac{p_1}{a} + \frac{p_2}{b}\right)$$

2. In this case, following a similar reasoning:

$$\bar{Q} = x_i \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

Then the cost function is:

$$C = \bar{Q}(p_1x_1 + p_2x_2 + \ldots + p_nx_n)$$

3. The same reasoning applies:

$$\bar{Q} = a_ix_i \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

Solving for x_i:

$$x_i = \frac{\bar{Q}}{a_i} \quad \text{for} \quad i = 1, 2, 3, \ldots, n$$

Replacing in the cost expression:

$$C = \bar{Q}\left(\frac{p_1x_1}{a_1} + \frac{p_2x_2}{a_2} + \ldots + \frac{p_nx_n}{a_n}\right)$$